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PD Control Algorithm:

$$\phi = K_1\gamma + K_2\dot{\gamma}$$

System Closed Loop Equation:

$$\ddot{\gamma} + \dot{\gamma}\left(K_2{\omega_x}^2\right) + \gamma\left({\omega_n}^2 + K_1{\omega_x}^2\right) = 0$$

Definition of γ :

$$\gamma \equiv \frac{J_B\theta + J_P\left(\theta+\phi\right)}{J_B + J_P} = \theta + \frac{J_P}{J_T}\phi$$

Evaluate System Parameters:

For simplicity, let $\omega_z = \omega_n$ from the above equations, where ω_z is the pendulum frequency of the uncontrolled (open loop) system.
Now let ω_n be the undamped natural frequency of the closed loop system with units of radians/sec.

ω_z can be found by locking the servo position such that $\phi = 0$ and manually applying a tilting force to the system. The resulting pendulum frequency in radians/sec is ω_z

ω_x can be found using the system equation of motion, tilt information from the camera system, geometry of the pendulum and the board, and known constants ω_z and ϕ.

System Equation of Motion*:

$$\ddot{\gamma} + \gamma{\omega_z}^2 = -\phi{\omega_x}^2$$

$j_P = \frac{-\theta_{MAX}}{\phi_{MAX}}$ when the servo is driven in a sinusoidal pattern. This is valid only when the acceleration due to gravity is zero. To achieve this, set the system on its side.

$$\omega_x{}^2 = \left(\frac{\theta}{\phi} + j_P\right)\omega_z{}^2$$ where $\frac{\theta}{\phi}$ is the ratio between the body and arm angles when the system is in equilibrium. The system must be set such that $\phi \neq 0$ and the system remains stationary.

We can measure the three constant parameters $\frac{\theta}{\phi}$, j_P, and ω_z.

From these we can find $\omega_X{}^2$.

Finding Appropriate K_1 and K_2 for PD Control:

The General 2nd Order Equation is:

$$\frac{1}{\omega_n{}^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = Ku(t)$$

Where ω_n is the undamped natural frequency of the closed loop system and ζ is the dimensionless damping ratio. Our system should be slightly underdamped, so $0.5 \leq \zeta \leq 1$.

Our 2nd order homogeneous differential equation describing our closed loop system can be rewritten as:

$$\frac{1}{\omega_z{}^2 + K_1\omega_x{}^2}\ddot{\gamma} + \frac{K_2\omega_x{}^2}{\omega_z{}^2 + K_1\omega_x{}^2}\dot{\gamma} + \gamma = 0$$

From these two equations, we see

$$\omega_n = \sqrt{\omega_z{}^2 + K_1\omega_x{}^2}$$ therefore $$\boxed{K_1 = \frac{\omega_n{}^2 - \omega_z{}^2}{\omega_x{}^2}}$$

$$\zeta = \frac{K_2 \omega_x^{\ 2} \omega_n}{2\left[\omega_z^{\ 2} + K_1 \omega_x^{\ 2}\right]} \quad \text{therefore} \quad K_2 = \frac{2\zeta\left[\omega_z^{\ 2} + K_1 \omega_x^{\ 2}\right]}{\omega_x^{\ 2} \omega_n}$$

Which simplifies to:

$$\boxed{K_2 = \frac{2\zeta\omega_n}{\omega_x^{\ 2}}}$$

Where ω_n is the chosen undamped natural frequency of the closed loop system and ζ is the chosen dimensionless damping ratio.
Because our control algorithm is

$$\phi = K_1\gamma + K_2\dot{\gamma}$$

and we now know both K_1 and K_2 and we can measure the current values for γ and $\dot{\gamma}$ we can find the necessary servo angle ϕ to keep the body level.

$$\gamma \equiv \frac{J_B\theta + J_P\left(\theta + \phi\right)}{J_B + J_P}$$

where ϕ above is the current servo angle value in radians. ϕ below is the commanded servo angle in degrees

$$ServoPosition = -\phi + 143$$